ARGUMENT OF BOUNDED ANALYTIC FUNCTIONS AND FROSTMAN'S TYPE CONDITIONS

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Abstract. We describe the growth of the naturally defined argument of a bounded analytic function in the unit disk in terms of the complete measure introduced by A.Grishin. As a consequence, we characterize the local behavior of a logarithm of an analytic function. We also find necessary and sufficient conditions for closeness of $\log f(z)$, $f \in H^\infty$, and the local concentration of the zeros of f.

1. Introduction

One of the basic theorems in complex analysis is the Argument principle, which states that if $f(z)$ is a meromorphic function inside and on some closed contour γ, with f having no zeros or poles on γ, then the increase of $\operatorname{Arg} f(z)$ along γ divided over 2π is equal to $N-P$, where N and P denote respectively the number of zeros and poles of $f(z)$ inside the contour γ. It seems reasonable to ask what can be said if the number of zeros (poles) of f is infinite. Obviously, the contour should contain a singular point and the increase of $\operatorname{Arg} f(z)$ along γ need not be bounded in this case. Theorem 2 of this paper can be considered as a generalization of the Argument principle for bounded analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We compare the growth of the naturally defined argument of a bounded analytic function F with the distribution of its complete measure in the sense of A.Grishin [11, 8].

Let us introduce some notation. We write $D(\zeta, \rho) = \{\xi \in \mathbb{C} : |\xi - \zeta| < \rho\}$ The symbols $C(\cdot)$ and $K(\cdot)$ stand for some positive constants depending on values in the parentheses, not necessarily the same in each occurence. Let H^∞ be the class of bounded analytic functions in $\mathbb{D}$. It is well-known [13, 6] that $f \in H^\infty$, $|f(z)| < C$, $z \in \mathbb{D}$, $C > 0$, can be represented in the form

$$f(z) = Cz^p \tilde{B}(z) g(z), \tag{1}$$

where p is nonnegative integer, $\tilde{B}$ is the Blaschke product constructed by the zeros of f,

$$\tilde{B}(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}(a_n - z)}{|a_n|(1 - z\overline{a}_n)} \equiv \prod_{n=1}^{\infty} \frac{b(z, a_n)}{|a_n|}, \quad a_n \neq 0, \ \sum_n (1 - |a_n|) < \infty, \tag{2}$$

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and g_ψ is an analytic function without zeros of the form

$$g_\psi(z) = \exp\left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\psi^*(t) + iC' \right\}, \tag{3}$$

where ψ^* is a non-decreasing function, and C' is a real constant.

We shall also consider the product

$$B(z) = \prod_{n=1}^{\infty} b(z, a_n) \tag{4}$$

which differs from $\tilde{B}(z)$ by a constant factor, provided that the Blaschke condition (2) holds. $B(z)$ converges almost everywhere to a finite limit $B(e^{i\theta})$ as z tends to $e^{i\theta}$ non-tangentially; moreover, $|\tilde{B}(e^{i\theta})| = 1$.

For a fixed θ_0 the following theorem of O. Frostman [6, 9] gives necessary and sufficient conditions for existence of the radial limit of $\tilde{B}(z)$.

Theorem A. *Necessary and sufficient that*

$$\lim_{r\uparrow 1} f(re^{i\theta_0}) = L \tag{5}$$

and $|L| = 1$ *for* $f = \tilde{B}$, *and every subproduct of* $\tilde{B}(z)$, *is that*

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0} - a_k|} < \infty. \tag{6}$$

If we drop the condition $|L| = 1$, then the theorem holds for B instead of $\tilde{B}$ as well.

Theorem A was generalized and complemented by many authors [2, 1, 5]. Usually one uses the condition

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|e^{i\theta_0} - a_k|^{1-\gamma}} < \infty \tag{7}$$

with $\gamma \le 0$ instead of (6). We note that if (7) holds with $\gamma \le 0$ and $|a_n - e^{i\theta_0}| < 1$, then there is only a finite number of zeros a_n in any Stolz angle with the vertex $e^{i\theta_0}$ where the Stolz angle with the vertex ζ is defined by

$$S_\sigma(\zeta) = \{\zeta \in \mathbb{D} : |1 - z\bar{\zeta}| \le \sigma(1-|z|)\}, \quad \sigma \ge 1,$$

provided that (7) is valid. We are interested in the case when (6) fails to hold, but (7) hold, when $\gamma \in (0, 1]$. The limit cases $\gamma = 1$ and $\gamma = 0$ correspond to the Blaschke condition and the Frostman condition, respectively. In this situation the zeros of B can be accumulated on the radius ending at $e^{i\theta_0}$, which is impossible when $\gamma \le 0$. Thus, $\arg B(z)$ should be defined carefully. If we want to obtain lower estimates for $|B(z)|$, $z \to e^{i\theta_0}$, $z \in \mathbb{D}$, we must exclude exceptional sets including the zero set.

Relations between conditions on the zeros of a Blaschke product B and the membership of $\arg B(e^{i\theta})$ in L^p spaces, $0 < p \le \infty$, were investigated in [19]. Criteria for boundedness of p-th integral means, $1 \le p < \infty$, of $\log |B|$ and $\log B$ were established in [18].

Since the proof of the necessity of Theorem A is based on estimates of the argument, one may ask whether it is possible to describe the zero distribution of a Blaschke product in terms of the behavior of $\arg B(z)$. A simple example shows that it is not sufficient to know the radial behavior of the argument.

Let (a_n) be an arbitrary Blaschke sequence with non-real elements. We define $c_{2n-1} = a_n$, $c_{2n} = \bar{a}_n$. Then

$$B(r) = \prod_{n=1}^{\infty} b(r, c_n) = \prod_{n=1}^{\infty} \frac{|a_n|^2 |a_n - r|^2}{|1 - r a_n|^2}, \quad 0 \le r < 1.$$

Thus,

$$\arg B(r) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \arg b(r, c_n) \equiv 0, \quad 0 \le r < 1.$$

But a situation is quite different if we consider the behavior of $\arg B(z)$ in a Stolz angle $S_\sigma(\zeta)$, $\zeta \in \partial\mathbb{D}$, $1 < \sigma < +\infty$, $S_\sigma = S_\sigma(1)$. Then we are able to describe the zero distribution, and even the distribution of the so-called complete measure in the sense of A. Grishin [11, 8].

Let $SH^\infty(\mathbb{D})$ be the class of subharmonic functions in $\mathbb{D}$ bounded from above. In particular, $\log |f| \in SH^\infty(\mathbb{D})$ if $f \in H^\infty$. Every function $u \in SH^\infty(\mathbb{D})$ which is harmonic in a neighborhood of the origin can be represented in the form (cf. [14, Ch.3.7])

$$u(z) = \int_{\mathbb{D}} \log \frac{|b(z,\zeta)|}{|\zeta|} d\mu_u(\zeta) - \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} d\psi(\zeta), \tag{8}$$

where μ_u is the Riesz measure of u [14], and ψ is a Borel measure on the unit circle. A *complete measure* λ_u *of* u *in the sense of Grishin* is defined [11, 8] by the boundary measure and the Riesz measure of $u(z)$. But, since [6]

$$\lim_{r \uparrow 1} \int_{\theta_1}^{\theta_2} \int_{\mathbb{D}} \log \frac{|b(re^{i\theta}, \zeta)|}{|\zeta|} d\mu_u(\zeta) d\theta = 0, \quad -\pi \le \theta_1 < \theta_2 \le \pi,$$

i.e. the boundary values of the first integral in (8) do not contribute to the boundary measure, we can define λ_u of a Borel set $M \subset \overline{\mathbb{D}}$ such that $M \cap \partial\mathbb{D}$ is measurable with respect to the Lebesgue measure on $\partial\mathbb{D}$ by

$$\lambda_u(M) = \int_{\mathbb{D} \cap M} (1 - |\zeta|)\, d\mu_u(\zeta) + \psi(M \cap \partial\mathbb{D}). \tag{9}$$

The measure $\lambda = \lambda_u$ has the following properties:

(1) λ is finite on $\overline{\mathbb{D}}$;
(2) λ is non-negative;
(3) λ is a zero measure outside $\overline{\mathbb{D}}$;

(4) $d\lambda\Big|_{\partial\mathbb{D}}(\zeta) = d\psi(\zeta)$;

(5) $d\lambda\Big|_{\mathbb{D}}(\zeta) = \ (1-|\zeta|)\, d\mu_u(\zeta)$.

If $u = \log|f|$, $f \in H^\infty$, then we shall write λ_f instead of $\lambda_{\log|f|}$. If $\tilde{B}$ is a Blaschke product of form (2), then $\lambda_{\tilde{B}}(M) = \sum_{a_n\in M}(1-|a_n|)$.

We shall say that g is a *divisor* of $f \in H^\infty$ if $g \in H^\infty$ and there exists an $h \in H^\infty$ such that $f = gh$. It is easy to see, that in this case we have $\lambda_g(M) + \lambda_h(M) = \lambda_f(M)$ for an arbitrary Borel subset M of $\overline{\mathbb{D}}$ such that $M \cap \partial\mathbb{D}$ is measurable.

The following generalization of Frostman's result on bounded functions is valid.

Theorem B (Lemma 3, [1])**.** *Let $F \in H^\infty$, and $\lambda_F(\{\zeta\}) = 0$ for some $\zeta \in \partial\mathbb{D}$. The following are equivalent.*

1)
$$\int_{\mathbb{D}} \frac{d\lambda_F(\xi)}{|\zeta-\xi|} < \infty.$$

2) *Every divisor of F has a radial limit at ζ.*

2. Main results and examples

Without loss of generality we can consider the local asymptotic behavior in a neighborhood of $\zeta = 1$ ($\theta_0 = 0$). Let $A(z,\xi) = \frac{1-|\xi|^2}{1-z\bar{\xi}}$, $\arg w$ be the principal branch of $\operatorname{Arg} w$.

Lemma 1. *Let $\xi \in \mathbb{D}$, $z \in \mathbb{D}\setminus\{\xi\}$. Then $|\arg b(z,\xi)| \le \pi \min\Big\{|A(z,\xi)|, 1\Big\}$.*

Consider the product $B(z)$ defined by (4). We make radial cuts $l_n = \{\zeta \in \mathbb{D} : \zeta = \tau a_n, \tau \ge 1\}$. The region $\mathbb{D}^* = \mathbb{D}\setminus\bigcup_{n=1}^\infty l_n$ contains no zeros of $B(z)$. Due to Lemma 1 we define (cf. [19]) a continuous branch

$$\log B(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \log b(z,a_n), \quad z \in \mathbb{D}^*,$$

$\arg B(z) \stackrel{\text{def}}{=} \Im \log B(z)$. In particular, we have $\log B(0) = 0$ and $\arg(B_1B_2) = \arg B_1 + \arg B_2$, where B_1, B_2 are Blaschke products. Later, in the proof of Lemma 1, we also define $\arg B(z)$ on the cuts except zeros. But the resulting function will not be continuous there.

In order to formulate our results we need some information on fractional derivatives [7, Chap.IX], [21, Chap.8]. For $h \in L(0,a)$ (integrable in the sense of Lebesgue on $(0,a)$) the fractional integral of Riemann-Liouville h_α of order $\alpha > 0$ is defined by the formula [12, 7, 21]

$$h_\alpha(r) = D^{-\alpha}h(r) = \frac{1}{\Gamma(\alpha)}\int_0^r (r-x)^{\alpha-1}h(x)\,dx, \quad r \in (0,a),$$

$$D^0 h(r) \equiv h(r), \quad D^{\alpha} h(r) = \frac{d^p}{dr^p}\{D^{-(p-\alpha)} h(r)\}, \quad \alpha \in (p-1, p], \ p \in \mathbb{N},$$

where $\Gamma(\alpha)$ is the Gamma function. The function h_α is continuous for $\alpha \geq 1$, and coincides with a primary function of the correspondent order when $\alpha \in \mathbb{N}$. We note that for $\alpha < 0$ the operator D^α is associative and commutative as a function of α. When writing $D^{-\alpha} f(z)$ we always mean that the operator is taken on the variable $r = |z|$.

Let $S^*_\sigma(\zeta) = S_\sigma(\zeta) \cap D(\zeta, \frac{1}{2})$. The following theorem yields a necessary and sufficient condition for the local growth of $\arg f$ in terms of the generalized Frostman's condition for the complete measure in the sense of Grishin of a bounded analytic function in the unit disk.

Theorem 2. *Let F be a bounded analytic function in $\mathbb{D}$, $0 \leq \gamma < 1$, $\zeta_0 \in \partial\mathbb{D}$. In order that for every divisor f of F and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, F) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma} \arg f(z)| < K, \tag{10}$$

it is necessary and sufficient that

$$\int_{\overline{\mathbb{D}}} \frac{d\lambda_F(\zeta)}{|\zeta_0 - \zeta|^{1-\gamma}} < \infty. \tag{11}$$

Remark 3. Since (10) must hold for every divisor f of F, (10) is equivalent to

$$\sup_{z \in S^*_\sigma(\zeta_0)} D^{-\gamma} |\arg f(z)| < K \tag{12}$$

for every divisor f and every $\sigma > 1$. In fact, we shall prove that (10) $\Rightarrow$ (11) $\Rightarrow$ (12). Since it is evident that (12) implies (10), this will prove Theorem 2.

Remark 4. As we shall see, in order that (11) hold it is sufficient that (10) holds for a finite number of divisors of a special form. Moreover, it is enough to require that

$$\overline{\lim_{z \to \zeta_0, z \in \Gamma_j}} |D^{-\gamma} \arg f(z)| < +\infty,$$

for two particular segments Γ_j ending at ζ_0, $\Gamma_j \subset \mathbb{D} \cup \{\zeta_0\}$, $j \in \{1, 2\}$.

Corollary 5. *Let B be a Blaschke product defined by (4), $0 \leq \gamma < 1$, $\zeta_0 \in \partial\mathbb{D}$. In order that for every subproduct B^* of B and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, B) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma} \arg B^*(z)| < K, \tag{13}$$

it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \frac{1 - |a_k|}{|\zeta_0 - a_k|^{1-\gamma}} < \infty. \tag{14}$$

Corollary 6. *Let $F \in H^\infty$, $0 \le \gamma < 1$. If (11) holds, then for every divisor f of F the function $\arg f(r)$ is bounded if $\gamma = 0$, and belongs to the convergence class of order γ if $\gamma \in (0,1)$, i.e.*

$$\int_0^1 (1-r)^{\gamma-1} |\arg f(r)| dr < +\infty.$$

Proof of Corollary 6. In fact, if $0 < \gamma < 1$, then

$$\sup_{0<r<1} D^{-\gamma} |\arg f(r)| = \sup_{0<r<1} \frac{1}{\Gamma(\gamma)} \int_0^r (r-x)^{\gamma-1} |\arg f(x)| dx \ge$$

$$\ge \sup_{0<r<1} \frac{1}{\Gamma(\gamma)} \int_0^r (1-x)^{\gamma-1} |\arg f(x)| dx = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-x)^{\gamma-1} |\arg f(x)| dx.$$

The case $\gamma = 0$ follows from Theorem A. □

Since for any $\sigma > 1$ we have $\mathbb{D} \subset \bigcup_{|\zeta|=1} S^*_\sigma(\zeta) \cup \overline{D}(0, \frac{1}{2})$, from Theorem 2 we get the following corollary.

Corollary 7. *Let F be a bounded analytic function in $\mathbb{D}$, $0 \le \gamma < 1$, and $\zeta_0 \in \partial\mathbb{D}$. Then for*

$$\sup_{z\in\mathbb{D}} |D^{-\gamma} \arg f(z)| < \infty$$

to hold, it is necessary and sufficient that

$$\sup_{\zeta_0 \in \partial\mathbb{D}} \int_{\overline{\mathbb{D}}} \frac{d\lambda_F(\zeta)}{|\zeta_0 - \zeta|^{1-\gamma}} < \infty.$$

Example 8. The analytic function

$$F(z) = \exp\Big\{-\frac{1+z}{1-z}\Big\}, \quad z \in \mathbb{D},$$

shows that the condition $\lambda_F(\{\zeta\}) = 0$ in Theorem B is essential. In fact, we have $\lambda_F(\zeta) = \delta(\zeta - 1)$ where $\delta(\zeta - 1)$ is the unit mass supported at $\zeta = 1$. The function F has the non-tangential limit 0 as $z \to 1$, $z \in \mathbb{D}$, but

$$\int_{\overline{\mathbb{D}}} \frac{d\lambda_F(\xi)}{|\xi - 1|^{1-\gamma}} = \infty, \quad \gamma < 1. \tag{15}$$

We have

$$\arg F(re^{i\varphi}) = \Im\Big\{-\frac{1+re^{i\varphi}}{1-re^{i\varphi}}\Big\} = -\frac{2r\sin\varphi}{|1-re^{i\varphi}|^2}.$$

It is clear that $\arg F(re^{i\beta(r-1)}) \to +\infty$ as $r \uparrow 1$ for any positive constant β. Theorem 2 yields that $D^{-\gamma} \arg F(z)$ is unbounded for any $\gamma < 1$, consequently

$$\arg F(z) \ne O\Big(\frac{1}{(1-|z|)^\gamma}\Big), \quad z \to 1, z \in S_\sigma, \sigma > 1, \gamma < 1.$$

The last relation follows from the fact that $h(r) = O((1-r)^{-\gamma})$ $(r \uparrow 1)$ implies $D^{-\gamma_1} h(r) = O(1)$ $(r \uparrow 1)$ provided $\gamma < \gamma_1 < 1$ (cf. Lemma 14 and the lemma from [4]).

Example 9. Let $\alpha \in [0,1)$,

$$\psi^*(t) = \begin{cases} t^{1-\alpha}, & t \in [0,\pi] \\ -|t|^{1-\alpha}, & t \in [-\pi, 0]. \end{cases}$$

Consider the function $g(z) = g_\psi(z)$ defined by (3), where $C' = 0$. Then g is analytic, bounded and has no zeros in $\mathbb{D}$. In this case $\lambda_g\Big|_{\mathbb{D}}$ is the zero measure, and $d\lambda_g(e^{it}) = d\psi(t)$, $t \in [-\pi, \pi]$. We have

$$\int_{\mathbb{D}} \frac{d\lambda_g(\zeta)}{|\zeta - 1|^{1-\gamma}} = \int_{-\pi}^{\pi} \frac{d\psi^*(t)}{|e^{it} - 1|^{1-\gamma}} = 2(1-\alpha) \int_0^{\pi} \frac{dt}{t^\alpha |e^{it} - 1|^{1-\gamma}}. \tag{16}$$

Since $|e^{it} - 1| \sim t$ as $t \downarrow 0$ the integral from (16) is convergent if and only if the integral $\int_0^\pi t^{-1-\alpha+\gamma} dt$ is convergent.

Thus, if $\gamma > \alpha$ we have

$$D^{-\gamma} \arg g_\psi(z) = O(1), \quad z \to 1, z \in S_\sigma, \sigma > 1.$$

In the limit case $\gamma = \alpha = 0$ one can show that

$$\arg g(r) \asymp \log \frac{1}{1-r}, \quad r \uparrow 1.$$

Now we consider the local behavior of the logarithm of a bounded function. Following C.N.Linden [17] we introduce characteristics of concentration of zeros. Let $n_z(h)$ be the number of zeros of an analytic function f in $\overline{D}(z, h(1-|z|))$,

$$N_z(h) = \sum_{|a_n - z| \le h(1-|z|)} \ln \frac{h(1-|z|)}{|z - a_n|} = \int\limits_0^{(1-|z|)h} \frac{n_z(s)}{s} ds.$$

These quantities are usually used for characterizing the local behavior of the modulus of an analytic function [15], [16].

Theorem 10. *Let $F \in H^\infty$, $0 \le \gamma < 1$, $0 < h < 1$, and $\zeta_0 \in \partial\mathbb{D}$. In order that for every divisor f of F and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, F) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma}(\log f(z) + N_z(h))| < K, \tag{17}$$

it is necessary and sufficient that

$$\int_{\mathbb{D}} \frac{d\lambda_F(\zeta)}{|\zeta_0 - \zeta|^{1-\gamma}} < \infty. \tag{18}$$

Corollary 11. *Let B be a Blaschke product defined by (2), $0 \le \gamma < 1$, $\zeta_0 \in \partial\mathbb{D}$, $0 < h < 1$. In order that for every subproduct B^* of B and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, B) > 0$ such that*

$$\sup_{z \in S_*(\zeta_0)} |D^{-\gamma}(\log B(z) + N_z(h))| < K$$

it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \frac{1-|a_k|}{|\zeta_0 - a_k|^{1-\gamma}} < \infty.$$

Statements of such type can be used for obtaining estimates for the minimum modulus of analytic and subharmonic functions ([15, 16, 17]), but we omit this topic here.

If F has no zeros, we easily obtain

Corollary 12. *Let $g \in H^\infty$ be of the form (3), $0 \le \gamma < 1$, $\zeta_0 \in \partial\mathbb{D}$. In order that for every divisor g^* of g and every $\sigma > 1$ there exist a constant $K = K(\gamma, \sigma, g) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma} \log g^*(z)| < K,$$

it is necessary and sufficient that

$$\int_{\partial\mathbb{D}} \frac{d\psi(\zeta)}{|\zeta_0 - \zeta|^{1-\gamma}} < \infty, \tag{19}$$

where ψ is the Stieltjes measure generated by ψ^*.

Let ψ and χ be Borel measures on $\partial\mathbb{D}$. We shall write that $\chi \prec \psi$ if $\chi(M) \le \psi(M)$ for an arbitrary Borel set $M \subset \partial\mathbb{D}$. Note that g_χ is a divisor of g_ψ if and only if $\chi \prec \psi$.

Applying Corollary 12 and Theorem 2 to the function $g_\psi(z) = \exp\{h_\psi(z)\}$ of form (3), we obtain

Theorem 13. *Let*

$$h_\psi(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\psi^*(t), \tag{20}$$

where ψ^* is a monotone function on $[-\pi, \pi]$. Let $0 \le \gamma < 1$, and $\zeta_0 \in \partial\mathbb{D}$. Let ψ be the Stieltjes measure generated by ψ^*. The following conditions are equivalent:

1) *For every Borel measure χ on $\partial\mathbb{D}$ such that $\chi \prec \psi$ and every $\sigma > 1$ there exists a constant $K = K(\gamma, \sigma, \psi) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma} h_\chi(z)| < K.$$

2) *For every Borel measure χ on ∂D such that $\chi \prec \psi$ and every $\sigma > 1$ there exists a constant $K = K(\gamma, \sigma, \psi) > 0$ such that*

$$\sup_{z \in S^*_\sigma(\zeta_0)} |D^{-\gamma} \Im h_\chi(z)| < K.$$

3) *Condition (19) holds.*

3. Proof of Theorem 2

We may assume that $\zeta_0 = 1$. We restrict ourself to the case $0 < \gamma < 1$. Let f be a divisor of F, and f of form (1). First, we consider $\arg B(z)$, and start with proof of Lemma 1.

Proof of Lemma 1. We consider the triangle with the vertices $A = z\bar{\xi}$, $B = |\xi|^2$, $C = 1$; $AB = 1 - |\xi|^2$, $BC = ||\xi|^2 - z\bar{\xi}|$, $AC = |1 - \bar{\xi}z|$. The quantity

$$\varphi_\xi = \arg b(z,\xi) = \arg \frac{|\xi|^2 - z\bar{\xi}}{1 - z\bar{\xi}}$$

is the value of the angle between the vectors $\vec{AB}$ and $\vec{AC}$. The cut $\{\zeta = \tau\xi : 1 \le \tau \le \frac{1}{|\zeta|}\}$ corresponds to BC. Thus, $|\varphi_\xi| < \pi$ if $z\bar{\xi} \notin BC$. For $z\bar{\xi} \in BC$, i.e. for z laying on the cut, we define by the semicontinuity $\varphi_\xi \overset{\text{def}}{=} -\pi$. Therefore, $\arg b(z,\xi)$ is defined in $\mathbb{D} \setminus \{\xi\}$ but, obviously, not continuous on the cut.

Let D_ξ be the disk constructed on AB as on the diameter. We consider two cases.

If $C = z\bar{\xi} \in D_\xi$, then $\pi/2 < |\varphi_\xi| \le \pi$ and $|z\bar{\xi} - 1| \le 1 - |\xi|^2$, i.e. $|A(z,\xi)| \ge 1$. Therefore $|\arg b(z,\xi)| \le \pi = \min\{\pi, |A(z,\xi)|\}$ as required.

If $z\bar{\xi} \notin D_\xi$, then $|\varphi_\xi| \le \pi/2$. Thus, $\varphi_\xi = \arcsin \frac{\Im b(z,\xi)}{|b(z,\xi)|}$. Since

$$\Im b(z,\xi) = -\Im A(z,\xi) = \Im(\bar{z}\xi)\frac{1 - |\xi|^2}{|1 - \bar{\xi}z|^2},$$

we have

$$|\varphi_\xi| = \left|\arcsin \frac{\Im(\bar{z}\xi)}{||\xi|^2 - z\bar{\xi}|}\frac{1 - |\xi|^2}{|1 - z\bar{\xi}|}\right| \le \tag{21}$$

$$\le \arcsin \min\left\{1, \frac{1 - |\xi|^2}{|1 - z\bar{\xi}|}\right\} \le \frac{\pi}{2}\min\{1, |A(z,\xi)|\}.$$

□

Lemma 14. *Let $0 \le \gamma < \alpha < \infty$. Then there exists a constant $C(\gamma,\alpha) > 0$ such that*

$$D^{-\gamma}\frac{1}{|1 - r\zeta|^\alpha} \le \frac{C(\gamma,\alpha)}{|1 - r\zeta|^{\alpha-\gamma}}, \quad \zeta \in \overline{\mathbb{D}}, 0 < r < 1 \tag{22}$$

Proof of Lemma 14. Let $\arg\zeta = \theta$. Then

$$|1 - x\zeta| \ge |1 - r\zeta|\cos(\theta/2), \quad 0 \le x \le r < 1. \tag{23}$$

In fact, geometric arguments yield that if $|r\zeta| \le \cos\theta$, then $|1 - x\zeta| \ge |1 - r\zeta|$. Otherwise, $\cos\theta < |r\zeta| < 1$, and we deduce

$$|1 - x\zeta| \ge |1 - e^{i\theta}\cos\theta| = |1 - e^{i\theta}|\cos(\theta/2) \ge |1 - r\zeta|\cos(\theta/2)$$

as required.

Without loss of generality we may assume that

$$|\theta| \le \pi/4, \quad \frac{1}{2} < r < 1, \quad 2|1-r\zeta| < r.$$

Using (23), we obtain

$$D^{-\gamma}\frac{1}{|1-r\zeta|^\alpha} = \frac{1}{\Gamma(\gamma)}\int_0^r \frac{(r-x)^{\gamma-1}}{|1-x\zeta|^\alpha}dx =$$
$$= \frac{1}{\Gamma(\gamma)}\left(\int_0^{r-2|1-r\zeta|} + \int_{r-2|1-r\zeta|}^r\right)\frac{(r-x)^{\gamma-1}}{|1-x\zeta|^\alpha}dx \le$$
$$\le \frac{1}{\Gamma(\gamma)}\left(\int_0^{r-2|1-r\zeta|}\frac{(r-x)^{\gamma-1}}{(1-x|\zeta|)^\alpha}dx + \int_{r-2|1-r\zeta|}^r \frac{(r-x)^{\gamma-1}}{|1-r\zeta|^\alpha \cos^\alpha\frac{\theta}{2}}dx\right) \le$$
$$\le \frac{1}{\Gamma(\gamma)}\left(\int_0^{r-2|1-r\zeta|}\frac{dx}{(1-x|\zeta|)^{1-\gamma+\alpha}} - \frac{(r-x)^\gamma}{\gamma|1-r\zeta|^\alpha\cos^\alpha\frac{\theta}{2}}\Big|_{r-2|1-\zeta r|}^r\right) =$$
$$= \frac{1}{\Gamma(\gamma)}\left(\frac{1}{(\alpha-\gamma)|\zeta|}\frac{1}{(1-x|\zeta|)^{\alpha-\gamma}}\Big|_0^{r-2|1-r\zeta|} + \frac{2^\gamma}{\gamma\cos^\alpha\theta/2|1-r\zeta|^{\alpha-\gamma}}\right) \le$$
$$\le \frac{1}{\Gamma(\gamma)}\left(\frac{2}{\alpha-\gamma}\frac{1}{(1-r+2|1-r\zeta||\zeta|)^{\alpha-\gamma}} + \frac{2^{\gamma+\alpha/2+1}}{\gamma|1-r\zeta|^{\alpha-\gamma}}\right) \le \frac{C(\gamma,\alpha)}{|1-r\zeta|^{\alpha-\gamma}}.$$

The lemma is proved. $\square$

In order to finish the proof of the sufficiency we need the following lemma ([10, Lemma 1]).

Lemma B. *Given $\sigma \ge 1$ there exists a constant $C(\sigma) > 0$ such that*

$$|1-\zeta| \le C(\sigma)|1-\bar z\zeta|, \quad \zeta \in \mathbb{D},\ z \in S_\sigma.$$

By Lemma 1 we have

$$|\arg B(z)| \le \pi\sum_{n=1}^\infty \frac{1-|a_n|^2}{|1-z\bar a_n|^2} \le \frac{C(F)}{1-r}.$$

Using Lemmas 1, 14, B and (11) we obtain for $z \in S_\sigma$

$$D^{-\gamma}|\arg B(z)| \le \sum_{n=1}^\infty D^{-\gamma}|\arg b(z,a_n)| \le$$
$$\le \pi\sum_{n=1}^\infty D^{-\gamma}\frac{1-|a_n|^2}{|1-z\bar a_n|} \le \pi C(\gamma)\sum_{n=1}^\infty \frac{1-|a_n|^2}{|1-z\bar a_n|^{1-\gamma}} \le$$
$$\le \pi C(\gamma,\sigma)\sum_{n=1}^\infty \frac{1-|a_n|^2}{|1-a_n|^{1-\gamma}} < C(\gamma,\sigma,F) < +\infty. \tag{24}$$

We now consider $\arg g(z)$. In view of (3) we have ($z = re^{i\varphi}$)

$$\arg g(z) = \Im\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\psi^*(t)\right\} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r\sin(\varphi-t)}{|e^{it}-z|^2}d\psi^*(t), \tag{25}$$

Using Lemmas 14 and B for $z \in S_\sigma$, we deduce

$$D^{-\gamma}|\arg g(z)| = \left|\frac{1}{\Gamma(\gamma)}\int_0^r (r-x)^{\gamma-1}dx\int_{-\pi}^{\pi}\frac{x\sin(\varphi-t)}{|e^{it}-xe^{i\varphi}|^2}d\psi^*(t)\right| \le$$
$$\le \int_{-\pi}^{\pi}\frac{|\sin(\varphi-t)|}{\Gamma(\gamma)}d\psi^*(t)\int_0^r\frac{(r-x)^{\gamma-1}}{|e^{it}-xe^{i\varphi}|^2}dx \le$$
$$\le C(\gamma)\int_{-\pi}^{\pi}\frac{|\sin(\varphi-t)|}{|e^{it}-re^{i\varphi}|^{2-\gamma}}d\psi^*(t) \le C(\gamma)\int_{-\pi}^{\pi}\frac{1}{|e^{it}-re^{i\varphi}|^{1-\gamma}}d\psi^*(t) \le$$
$$\le C(\gamma,\sigma)\int_{-\pi}^{\pi}\frac{1}{|e^{it}-1|^{1-\gamma}}d\psi^*(t).$$

The sufficiency is proved.

Necessity. First, we consider the subproduct B^* of B constructed by the zeros a_n satisfying $\Im a_n \ge 0$, $|1-a_n| \le \frac{1}{3}$. We denote such a_n by a_n^*.

Let $z = re^{i\varphi}$ satisfy $\arg(1-z) = \pi/4$, $\zeta \in [0,z]$, $\zeta = \rho$. In particular, $\Im\zeta < 0$. Then

$$\Im(a_n^*\bar\zeta) = -\Im\zeta\Re a_n^* + \Im a_n^*\Re\zeta \ge 0,$$

and consequently (see (21))

$$\arg b(\zeta, a_n^*) \ge \arcsin\frac{\Im(\bar\zeta a_n^*)(1-|a_n^*|^2)}{||a_n^*|^2-\zeta\bar a_n^*||1-\bar a_n^*\zeta|} \ge 0.$$

By our assumption

$$C \ge D^{-\gamma}\arg B^*(z) = \sum_n D^{-\gamma}\arg b(z,a_n^*) \ge$$
$$\ge \sum_n\int_0^r (r-t)^{\gamma-1}\arcsin\frac{\Im(te^{-i\varphi}a_n^*)(1-|a_n^*|^2)}{||a_n^*|^2-te^{i\varphi}\bar a_n^*||1-\bar a_n^* te^{-i\varphi}|}dt. \tag{26}$$

For every a_n^* satisfying $1-|a_n^*| \ge 2(1-r)$ and $\zeta \in [0,z]$ such that

$$|1-a_n^*| \le r-\rho \le 2|1-a_n^*|$$

we have

$$\rho \ge r - 2|1-a_n^*| \ge r - \frac{2}{3} > \frac{1}{4}, \quad r \uparrow 1.$$

Thus,

$$|\Im\zeta| \ge |\Im z|/4 \ge (1-r)/4.$$

Hence,

$$\Im(a_n^*\bar\zeta) \ge -\Im\zeta\Re a_n^* \ge \frac{\Re a_n^*}{4}(1-r). \tag{27}$$

Similarly,

$$\Im(a_n^*\bar\zeta) \ge \Re\zeta\Im a_n^* \ge \frac{\Re z}{2}|1-a_n^*|, \quad a_n^* \notin S_2. \tag{28}$$

Further,

$$|a_n^*-\zeta| \le |1-a_n^*| + 2|1-|\zeta|| = |1-a_n^*| + 2(r-|\zeta|+1-r) \le$$

$$\le |1-a_n^*| + 2(2+\frac{1}{2})|1-a_n^*| = 6|1-a_n^*|, \tag{29}$$

$$|1-\bar a_n^*\zeta| \le |1-\bar a_n^*| + |\bar a_n^* - \bar a_n^*\zeta| \le 6|1-a_n^*|. \tag{30}$$

Thus, for $a_n^* \in S_2$ using (27), (29), and (30) we have

$$\int_0^r (r-t)^{\gamma-1} \arcsin \frac{\Im(\bar a_n^* te^{i\varphi})(1-|a_n^*|^2)}{||a_n^*|^2 - te^{i\varphi}\bar a_n^*||1-\bar a_n^* te^{i\varphi}|} dt \ge$$

$$\ge \int_0^r \frac{(r-t)^{\gamma-1}\Re a_n^*(1-t)(1-|a_n^*|^2)}{144|1-a_n^*|^2|a_n^*|} dt \ge$$

$$\ge C(\gamma) \int\limits_{r-2|1-a_n^*|}^{r-|1-a_n^*|} \frac{(r-t)^\gamma \Re a_n^*}{|a_n^*|(1-|a_n^*|)} dt \ge C(\gamma)\frac{1}{1-|a_n^*|} \int\limits_{r-2|1-a_n^*|}^{r-|1-a_n^*|} (r-t)^\gamma \, dt \ge$$

$$\ge C(\gamma)(1-|a_n^*|)^\gamma. \tag{31}$$

If $a_n^* \in \mathbb{D} \setminus S_2$, then using (28)–(30) we obtain

$$\int_0^r (r-t)^{\gamma-1} \arcsin \frac{\Im(\bar a_n^* te^{i\varphi})(1-|a_n^*|^2)}{||a_n^*|^2 - te^{i\varphi}\bar a_n^*||1-\bar a_n^* te^{i\varphi}|} dt \ge$$

$$\ge \int_0^r \frac{(r-t)^{\gamma-1}\Re z|1-a_n^*|(1-|a_n^*|^2)}{72|1-a_n^*|^2|a_n^*|} dt \ge$$

$$\ge C(\gamma) \int\limits_{r-2|1-a_n^*|}^{r-|1-a_n^*|} \frac{(r-t)^{\gamma-1}\Re z|1-|a_n^*||}{|a_n^*||1-a_n^*|} dt \ge$$

$$\ge C\frac{1-|a_n^*|}{|1-a_n^*|} \int\limits_{r-2|1-a_n^*|}^{r-|1-a_n^*|} (r-t)^{\gamma-1} \, dt \ge C\frac{1-|a_n^*|}{|1-a_n^*|^{1-\gamma}}. \tag{32}$$

Hence,

$$C > D^{-\gamma} \arg B^*(z) = \sum_n D^{-\gamma} \arg b(z, a_n^*) > C \sum_{|a_n^*| \le 1-2(1-r)} \frac{1-|a_n^*|}{|1-a_n^*|^{1-\gamma}}.$$

Since the constants C are independent of r, tending $r \uparrow 1$ we get the statement of the necessity for $\arg B^*$, and consequently for $\arg B$.

Now, we have to estimate $D^\gamma(\arg g_\psi)$ from below. Let ψ_1 be the restricted function of ψ^* on $[0,\pi/2]$. Let $\arg(1-z)=\frac{\pi}{4}$. Then

$$D^{-\gamma}\Im g_{\psi_1}(z)=\frac{1}{\Gamma(\gamma)}\int_{-\pi}^{\pi}\int_0^r \frac{(r-\rho)^{\gamma-1}\sin(t-\varphi)}{|\rho e^{i\varphi}-e^{it}|^2}d\rho d\psi_1(t)=$$
$$=\frac{1}{\Gamma(\gamma)}\int_0^{\pi/2}\sin(t-\varphi)d\psi^*(t)\int_0^r\frac{(r-\rho)^{\gamma-1}}{|\rho e^{i\varphi}-e^{it}|^2}d\rho.$$

In order to estimate the inner integral we may assume that $r>2|z-e^{it}|$ without loss of generality. For $|z-e^{it}|\le r-\rho\le 2|z-e^{it}|$ we have

$$|\rho e^{i\varphi}-e^{it}|\le|z-\rho e^{i\varphi}|+|z-e^{it}|\le$$

$$\le(1+o(1))|r-\rho|+|z-e^{it}|\le 4|z-e^{it}|,\quad r\uparrow 1.$$

Moreover, since $\arg z\sim r-1$, we have $t-\varphi\ge(1+o(1))(1-r)$ as $r\uparrow 1$. Then,

$$|z-e^{it}|=|r-e^{i(t-\varphi)}|\le 1-r+1-\cos(\varphi-t)+\sin(\varphi-t)\le$$
$$\le(1+o(1))\sin(1-r)+2\sin^2\frac{t-\varphi}{2}+\sin(t-\varphi)\le$$
$$\le(4+o(1))\sin(t-\varphi),\quad r\uparrow 1.$$

Using the latter estimates we deduce

$$C\ge D^{-\gamma}\Im g(z)\ge\frac{1}{\Gamma(\gamma)}\int_0^{\pi/2}\sin(t-\varphi)d\psi^*(t)\int_{r-2|z-e^{it}|}^{r-|z-e^{it}|}\frac{(r-\rho)^{\gamma-1}}{|\rho e^{i\varphi}-e^{it}|^2}d\rho\ge$$
$$\ge\int_0^{\pi/2}\frac{\sin(t-\varphi)}{16|z-e^{it}|^2}d\psi^*(t)\int_{r-2|z-e^{it}|}^{r-|z-e^{it}|}(r-\rho)^{\gamma-1}d\rho\ge$$
$$\ge C(\gamma)\int_0^{\pi/2}\frac{\sin(t-\varphi)|z-e^{it}|^\gamma}{|z-e^{it}|^2}d\psi^*(t)\ge C(\gamma)\int_0^{\pi/2}\frac{d\psi^*(t)}{|z-e^{it}|^{1-\gamma}}.$$

Tending r to 1 and using Fatou's lemma we conclude that

$$C\ge C(\gamma)\int_0^{\pi/2}\frac{d\psi^*(t)}{|1-e^{it}|^{1-\gamma}}.$$

Similarly, it can be shown that $\int_{-\pi/2}^0\frac{d\psi(t)}{|1-e^{it}|^{1-\gamma}}<C$, and consequently,

$$\int_{-\pi}^{\pi}\frac{d\psi(t)}{|1-e^{it}|^{1-\gamma}}<C.$$

Theorem 2 is proved.

4. Proof of Theorem 10 and final remarks

Proof of Theorem 10. The necessity of the theorem follows from Theorem 2.

Sufficiency. Let f be a divisor of F. Without loss of generality we may assume that $f = Bg$, where B and g are defined as above. Let $L(z, h, f) = \log f(z) + N_z(h)$. We have

$$\Re L(z,h,f) = \Re L(z,h,B) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re\frac{e^{it}+z}{e^{it}-z}d\psi^*(t) =$$

$$= \sum_{|a_n-z|\le h(1-r)} \ln\Big|\frac{a_n h(1-r)}{1-z\bar a_n}\Big| + \sum_{|a_n-z|>h(1-r)} \ln\Big|\frac{a_n(z-a_n)}{1-z\bar a_n}\Big| -$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it}-z|^2}d\psi^*(t) \le 0. \tag{33}$$

Let us estimate $\Re L(z,h,f)$ from the below. For $|a_n - z| \le h(1-r)$ we have

$$|1 - z\bar a_n| = |1-|z|^2 + z(\overline{z-a_n})| \le 1 - r^2 + rh(1-r) \le (2+h)(1-r),$$

and

$$|a_n| \ge r - h(1-r) \ge \frac{1-h}{2}, \quad r \ge \frac12. \tag{34}$$

Hence,

$$\sum_{|a_n-z|\le h(1-r)} \ln\Big|\frac{a_n h(1-r)}{1-z\bar a_n}\Big| \ge \sum_{|a_n-z|\le h(1-r)} \ln\frac{|a_n|h}{2+h} \ge$$

$$\ge -C(h)\sum_{|a_n-z|\le h(1-r)} |A(z,a_n)|, \quad r\ge \frac12, \tag{35}$$

because $C_1 \le |A(z,a_n)| \le C_2$ if $|a_n - z| \le h(1-r)$.

On the other hand, (see [20, p.13])

$$\sum_{|A(z,a_n)|<\frac12} -\ln|b(z,a_n)| \le 2\sum_{|A(z,a_n)|<\frac12} |A(z,a_n)|. \tag{36}$$

It is known that a pseudohyperbolic disk $\mathcal{D}(z,s) = \Big\{\zeta : \Big|\frac{z-\zeta}{1-z\zeta}\Big| < s\Big\}$ is the disk $D(z^*, \rho_z(s))$, where

$$z^* = \frac{(1-s^2)z}{1-s^2|z|^2}, \quad \rho_z(s) = \frac{(1-|z|^2)s}{1-s^2|z|^2}.$$

We are going to prove that

$$\mathcal{D}\Big(z, \frac{h}{2+h}\Big) \subset D(z, (1-|z|)h). \tag{37}$$

It is sufficient to show that $|z^*-z|+\rho_z(s)\le h(1-|z|)$ for $s\le h/(2+h)$. We have ($|z|=r$)

$$|z^*-z|+\rho_z(s)=\frac{(1-r^2)(rs^2+s)}{1-s^2r^2}\le\frac{2(1-r)s}{1-s}.$$

Thus, we arrive to the inequality $2s\le h(1-s)$, which is equivalent to $-1\le s\le\frac{h}{2+h}$. Inclusion (37) is proved. Therefore, for $a_n\notin D(z,h(1-r))$ we have

$$-\ln|b(z,a_n)|\le\ln\frac{2+h}{h|\bar a_n|}.$$

Hence, using (34)

$$\begin{aligned}\sum_{\substack{|A(z,a_n)|\ge\frac12\\|a_n-z|>h(1-r)}}-\ln|b(z,a_n)|&\le 2\sum_{\substack{|A(z,a_n)|\ge\frac12\\|a_n-z|>h(1-r)}}\ln\frac{2+h}{|\bar a_n|h}\le\\ &\le 4\ln\frac{4+2h}{h(1-h)}\sum_{\substack{|A(z,a_n)|\ge\frac12\\|a_n-z|>h(1-r)}}|A(z,a_n)|.\end{aligned}\tag{38}$$

It follows from (33)–(38) that

$$\Re L(z,h,B)\ge-C(h)\sum_n|A(z,a_n)|.$$

Hence, as in the proof of the sufficiency of Theorem 2 (see (3)) we deduce for $z\in S_\sigma$

$$D^{-\gamma}\Re L(z,h,B)\ge-C(h,\gamma)\sum_n\frac{1-|a_n|^2}{|1-\bar a_nz|^{1-\gamma}}\ge-C(h,\gamma,\sigma,B).\tag{39}$$

Further,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it}-z|^2}d\psi^*(t)\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\psi^*(t)}{|e^{it}-z|}.$$

Applying Lemma 2 for z laying in the Stolz angle S_σ, we obtain

$$\begin{aligned}D^{-\gamma}\left|\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it}-z|^2}d\psi^*(t)\right|&\le\int_{-\pi}^{\pi}D^{-\gamma}\Big(\frac{1}{|e^{it}-z|}\Big)d\psi^*(t)\le\\ &\le C(\gamma)\int_{-\pi}^{\pi}\frac{d\psi^*(t)}{|e^{it}-z|^{1-\gamma}}\le C(\gamma,\sigma)\int_{-\pi}^{\pi}\frac{d\psi(t)}{|e^{it}-1|^{1-\gamma}}<\infty.\end{aligned}\tag{40}$$

Together with (39) this yields $D^{-\gamma}\Re L(z,h,f)\ge-C$. And, in view of (33) we, finally, have $|D^{-\gamma}\Re L(z,h,F)|\le C$.

It remains to apply Theorem 2. Theorem 10 is proved. □

Remark 15. Frostman type condition (11) can be rewritten in terms of the modulus of continuity of the complete measure. Let $\lambda_F(\zeta,\tau) \stackrel{\text{def}}{=} \lambda_F(\overline{D(\zeta,\tau)})$. Then (11) is equivalent to

$$\int_0^2 \frac{d\lambda_F(\zeta_0,\tau)}{\tau^{1-\gamma}} < +\infty \ \text{ or } \int_0^2 \frac{d\omega(\tau;\zeta_0,\lambda_F)}{\tau^{1-\gamma}} < +\infty$$

where $\omega(\tau;\zeta_0,\lambda_F)$ is the modulus of continuity of the measure λ_F at the point ζ_0.

From this point of view it is interesting to compare Theorem 13 with results from [4], where necessary and sufficient conditions for growth of the maximum modulus and the maximum of the real part of h_ψ is established in terms of the modulus of continuity of the function ψ^*. Similar results for L^p-metrics are obtained in [3].

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